PHYS 320 ANALYTICAL MECHANICS

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The Simple Harmonic Oscillator

- Differential equation:

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \omega_o \equiv \sqrt{\frac{k}{m}} = \text{ natural frequency}$$

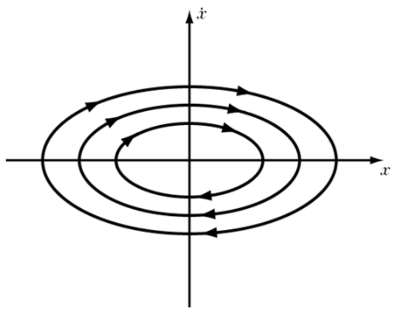
- Solutions:

$$x(t) = C_1 e^{+i\omega_o t} + C_2 e^{-i\omega_o t} = B_1 \cos(\omega_o t) + B_2 \sin(\omega_o t) = A\cos(\omega_o t - \delta)$$

Phase plots

- For a 1-D oscillator, the motion is completely specified by two quantities: $x(t)$ and $\dot{x}(t)$
- Note that $E_{mech} = K + U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$

$$= \frac{1}{2} m (-A\omega_o \sin(\omega_o t - \phi))^2 + \frac{1}{2} k (A\cos(\omega_o t - \phi))^2$$



$$\frac{x(t)^2}{A^2} + \frac{\dot{x}(t)^2}{\omega_o^2 A^2} = 1$$

Family of ellipses!

Phase plots with MAPLE

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> restart;
> with(plots):
> x:=A*cos(omega*t+phi);
> v:=diff(x,t);
> phi:=0;A:=1;omega:=1/2;
> plot([x,v,t=0..4*Pi],labels=["x(t)","v(t)"],scaling=constrained);
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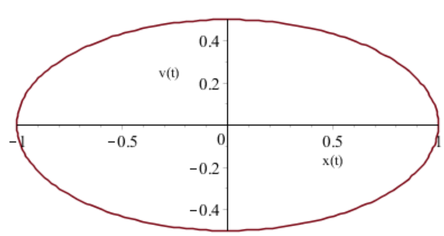


Damped Oscillations (linear damping)

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$ with trial solution $x(t)=Ae^{qt}$

yields auxiliary equation

$$mq^2+cq+k=0 \quad \Rightarrow \quad q=\frac{-c\pm\sqrt{c^2-4mk}}{2m}$$

∃ Three possibilities:

- $c^2 > 4mk$ overdamped
- $c^2 = 4mk$ critically damped
- $c^2 < 4mk$ underdamped

Damped Oscillations (linear damping)

auxiliary equation:

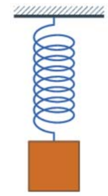
$$q=\frac{-c\pm\sqrt{c^2-4mk}}{2m}=-\gamma\pm\sqrt{\gamma^2-\omega_o^2}$$

where $\gamma\equiv\frac{c}{2m}=$ damping parameter

(note that Taylor uses β)

and $\omega_o\equiv\sqrt{\frac{k}{m}}=$ natural frequency

Damped Oscillations (linear damping)



$$c^2 < 4mk \quad (\gamma < \omega_o) \quad \text{underdamped}$$

yields actual oscillations!

$$x(t) = C_+ e^{-\gamma t} e^{+i\sqrt{\omega_o^2 - \gamma^2}\, t} + C_- e^{-\gamma t} e^{-i\sqrt{\omega_o^2 - \gamma^2}\, t}$$

$$= B_1 e^{-\gamma t}\cos(\omega_d t) + B_2 e^{-\gamma t}\sin(\omega_d t)$$

$$= A e^{-\gamma t}\cos(\omega_d t - \phi)$$

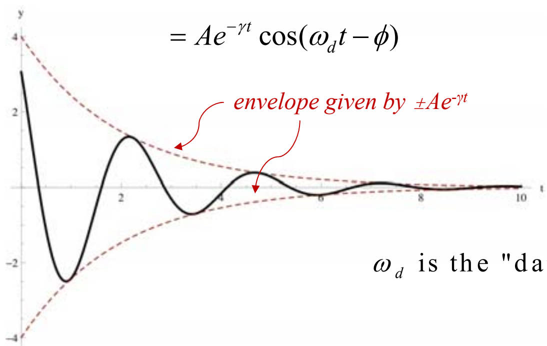
where

$$\omega_d^2 \equiv \omega_o^2 - \gamma^2$$

$$\phi \equiv \tan^{-1}(B_2 / B_1)$$

$$A^2 \equiv B_1^2 + B_2^2$$

$$C_\pm \equiv (B_1 \mp i B_2)/2$$



$$\omega_o \equiv \sqrt{\frac{k}{m}}$$

ω_d is the "damped frequency"

(Note Taylor uses ω_1)

$$\gamma \equiv \frac{c}{2m}$$